

A Model for BCS-Type Correlations in Superscaling

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Using ideas from BCS descriptions of systems of fermions, a covariant extension of the relativistic Fermi gas model is presented as a way to incorporate correlation effects in nuclei. The model is developed for the BCS nuclear ground state and for final states consisting of a single plane-wave nucleon plus a BCS recoiling daughter nucleus. The nuclear spectral function is obtained and from it the superscaling function for use in treating high-energy quasielastic electroweak processes. Interestingly, this model has the capability to yield the asymmetric tail seen in the experimental scaling function.

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I. INTRODUCTION

The issue of superscaling in electroweak interactions with atomic nuclei has been the focus in recent studies [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] because of its intrinsic interest and because it offers a way to obtain reliable charge-changing and neutral-current neutrino-nucleus cross sections. The latter are usually essential when addressing fundamental neutrino properties, specifically neutrino masses and the neutrino oscillations that result from those masses. A summary of the basic concepts involved in superscaling analyses is provided below in Section II.

Owing to the complexity of nuclear dynamics it is not obvious that the nuclear response to an electroweak field superscales; indeed for this to occur the single-nucleon cross section should factorize, which is almost never completely true. Certainly in extreme on-shell independent-particle models such as the Relativistic Fermi Gas (RFG) this is valid, although in general factorization is seen to be violated to some degree, since the nucleons in the nucleus are not on-shell and the independent-particle model is only an approximation. For instance, in other studies done within the context of the shell model (for instance, [10, 11]) one sees breaking of factorization from off-shell effects, typically at the few percent level for the kinematics of interest when applying the ideas of superscaling, namely, momentum transfers of order 1 GeV/c and energy transfers near the quasielastic peak position. Clearly, if collective effects (for instance, giants resonances) are important, the independent-particle description breaks down and thus superscaling should not be expected to hold in the low momentum and energy transfer domain where such effects occur. Finally, when meson-exchange currents play a significant role the superscaling violations that may or may not occur require detailed study [18, 19, 20, 21].

In the present work we shall not address these general issues, but instead shall focus on an extension of the RFG approach, namely, we shall include only on-shell nucleons in an independent-particle model where factorization is exact and put our efforts into going beyond the degenerate description provided by the extreme RFG model. To do this we resort to a model inspired by the BCS theory of condensed matter physics (see, *e.g.*, [22]) in which electron Cooper pairs are correlated via an attractive interaction mediated by the lattice. In nuclei correlations among pairs of nucleons also occur [23], in this case mediated by NN interactions including both the long-range attractive residual interaction (the pairing force) and also the strong repulsive force that occurs at short distances. However, and remarkably, the BCS wave function can be shown to hold valid (in the thermodynamic limit) in both instances in the sense of being an eigenstate of the well-known pairing Hamiltonian independently of the sign of the interaction (of course in the attractive case the BCS state provides a minimum for the system's energy, in the repulsive one it yields instead a maximum).

Hence we shall deal with the problem of assessing the impact of correlations on the superscaling function employing the BCS formalism with appropriate modifications, such as retention of covariance, to adapt it to the high-energy physics of atomic nuclei. This is addressed in Section III after having briefly outlined in Section II the basics of the superscaling approach for the longitudinal nuclear response R_L , namely the part of the total inclusive electroweak response that is believed to superscale the best.

The numerical results obtained with our approach are presented in Section IV. To summarize briefly our findings, we shall see that, likely because in the BCS spirit we limit ourselves to an independent quasi-particle description of nuclear matter, scaling of the first kind (independence of momentum transfers q) appears to occur not only at the quasielastic peak (QEP), but also at both lower and higher energy transfers ω . We find that the onset of first-kind

scaling already occurs at momentum transfers of order 500 MeV/c, whereas away from the QEP the onset only occurs at quite large momentum transfers (of the order of 2 GeV/c). Furthermore, interestingly the shape of the superscaling function turns out to be *non-symmetric* around the QEP, being larger to the right and smaller to the left of it, namely, in agreement with experiment and thus lending support to our approach. However, in our model when in the so-called scaling region (below the QEP) first-kind scaling is reached as a function of q from below, which is not what is experimentally found. Finally, scaling of the second kind (independence of nuclear species) is also explored in Section IV and shown to be relatively well satisfied, given that an appropriate momentum scale is chosen for each nuclear species.

In the Conclusions (Section V) we briefly discuss the significance and limitations of our model.

II. THE LONGITUDINAL RESPONSE FUNCTION

As anticipated in the Introduction we shall concentrate on the longitudinal EM channel; all other electroweak responses can be developed using similar arguments to those presented in the following. We start by briefly summarizing the basics of the superscaling approach, which has been discussed in depth in previous work [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Here we employ the same notation and formalism as in the paper by Cenni-Donnelly-Molinari [3] (to be referred to as CDM).

Within the framework of the Plane-Wave Impulse Approximation (PWIA) the longitudinal response function of a nucleus to an external electroweak field bringing three-momentum \mathbf{q} and energy ω into the system reads

$$R_L(q, \omega) = \iint_{\Sigma} dp d\mathcal{E} p^2 \frac{E_N}{pq} 2\pi S(p, \mathcal{E}) \frac{m_N}{E_p} \frac{m_N}{E_N} \mathcal{R}_L(q, \omega, p, \mathcal{E}), \quad (1)$$

where $E_p = \sqrt{\mathbf{p}^2 + m_N^2}$ and $E_N = \sqrt{(\mathbf{p} + \mathbf{q})^2 + m_N^2}$ are the energies of the initial (struck) and final (ejected) nucleons, respectively, with m_N the nucleon mass. Indeed in PWIA it is assumed that only one vector boson is exchanged between the probe and the nucleus and that this one is absorbed by a single nucleon. The probability of finding one nucleon in the system is provided by the spectral function of the system, $S(p, \mathcal{E})$, which depends on the missing momentum p and an energy \mathcal{E} , the latter being essentially the missing energy minus the separation energy and given explicitly by the excitation energy of the residual nucleus in the reference frame where it moves with momentum $-\mathbf{p}$. The factor \mathcal{R}_L in Eq. (1) is the longitudinal single-nucleon response which is in general half-off-shell and hence a function not only of q and ω , but also of the energy and momentum of the off-shell struck nucleon, or equivalently of p and \mathcal{E} . In the models being considered in the present study the struck nucleon is in fact on-shell and so \mathcal{R}_L becomes simply the longitudinal response of a moving free nucleon:

$$\mathcal{R}_L = \frac{\kappa^2}{\tau} \{G_E^2(\tau) + W_2(\tau)\chi^2\}, \quad (2)$$

where

$$W_2(\tau) = \frac{1}{1 + \tau} [G_E^2(\tau) + \tau G_M^2(\tau)], \quad (3)$$

expressed in terms of the electric G_E and magnetic G_M Sachs form factors of the nucleon and $\chi = (p/m_N) \sin \theta$, where θ is the angle between \mathbf{p} and \mathbf{q} . As in CDM in Eqs. (2) and (3) the dimensionless variables

$$\kappa = \frac{q}{2m_N}, \quad \lambda = \frac{\omega}{2m_N} \quad \text{and} \quad \tau = \kappa^2 - \lambda^2 \quad (4)$$

are employed.

Equation (1) connects the semi-inclusive $(e, e'N)$ reaction with the inclusive (e, e') process assuming that the outgoing nucleon no longer interacts with the residual $(A - 1)$ nucleus (absence of final-state interactions). That equation expresses the assumption that the inclusive cross section is to be obtained by integrating the semi-inclusive cross section, summing over struck protons and neutrons. The boundaries of the integration domain Σ in the (\mathcal{E}, p) plane are found through the energy conservation relation (see CDM) and read

$$\mathcal{E}^{\pm}(p) = \omega - E_s + m_N - \sqrt{(p \pm q)^2 + m_N^2} - T_0^{(A-1)}, \quad (5)$$

where $\mathcal{E}^+ \leq \mathcal{E} \leq \mathcal{E}^-$ and \mathcal{E}^- is bounded according to

$$0 \leq \mathcal{E}^- \leq \omega - E_s - \left[\sqrt{q^2 + (M_{A-1} + m_N)^2} - (M_{A-1} + m_N) \right]. \quad (6)$$

In the above

$$E_s = M_{A-1} + m_N - M_A \quad (7)$$

is the separation energy of a nucleon from the ground state of the nucleus A (of mass M_A) leaving the residual $(A-1)$ nucleus (of mass M_{A-1}) in its ground state as well and

$$T_0^{(A-1)} = \sqrt{p^2 + M_{A-1}^2} - M_{A-1} \quad (8)$$

is the very small recoiling nucleus kinetic energy, which is often neglected in Eq. (5). Careful investigations of Eq. (1) have shown that the single-nucleon response function is quite gently varying over the integration domain Σ in the (\mathcal{E}, p) plane, particularly if one limits the focus only to regions where the spectral function $S(p, \mathcal{E})$ plays a significant role (see CDM); accordingly it can be extracted from the integral in Eq. (1) and divided out, thereby incurring only typically few percent errors. This is the point at which the factorization discussed in the Introduction is invoked.

Finally, in this study we shall confine ourselves to dealing with infinite, homogeneous systems, the simplest among them being the RFG model in which the dynamics are controlled by just one parameter, the Fermi momentum k_F . To attain superscaling it then turns out to be convenient to multiply Eq. (1) by a further factor that, in the RFG case, is proportional to the momentum transfer and to k_F . Indeed, as we shall see, this leads to a function which loses any dependence on both k_F and q , namely, one has superscaling for the RFG in the non-Pauli-blocked regime. It remains to be seen what happens in the BCS model, *i.e.*, in the presence of correlations.

III. THE SPECTRAL FUNCTION

As discussed in the previous section the key tool for exploring superscaling is the nuclear spectral function $S(p, \mathcal{E})$, defined as follows

$$S(p, \mathcal{E}) = \langle \Psi_0 | a_{\mathbf{p}\uparrow}^\dagger \delta(\mathcal{E} - (\hat{H} - E_0^{(A-1)})) a_{\mathbf{p}\uparrow} | \Psi_0 \rangle \frac{V_A}{A(2\pi)^3} \quad (9)$$

in terms of matrix elements of contributions involving the standard fermion creation and annihilation operators. In the above $|\Psi_0\rangle$ and \hat{H} are the ground state and hamiltonian of the system, respectively, and V_A is the (large) volume enclosing the system.

A. The RFG model

The RFG spectral function is easily computed when one recalls (see CDM) that it has a negative separation energy

$$(E_s)_{RFG} = -T_F, \quad (10)$$

$T_F = \sqrt{k_F^2 + m_N^2} - m_N$ being the Fermi kinetic energy. One obtains

$$S^{RFG}(p, \mathcal{E}) = \sum_{spin, isospin} \theta(k_F - p) \delta(\mathcal{E} - T_F + T_p) \frac{V_A}{A(2\pi)^3}, \quad (11)$$

where the nucleon kinetic energy $T_p = \sqrt{p^2 + m_N^2} - m_N$ has been introduced, and observes that it is nonzero in the (\mathcal{E}, p) plane only on the curve

$$\mathcal{E} = \sqrt{k_F^2 + m_N^2} - \sqrt{p^2 + m_N^2} \quad (12)$$

and is normalized to unity, as is easy to check by integrating Eq. (11) over \mathbf{p} and the non-negative variable \mathcal{E} . An integration only over \mathcal{E} yields the system's momentum distribution normalized to the Fermi sphere, namely

$$\int_0^\infty d\mathcal{E} S^{RFG}(p, \mathcal{E}) = \frac{\theta(k_F - p)}{\frac{4}{3}\pi k_F^3}. \quad (13)$$

Inserting Eq. (11) into Eq. (1), after having pulled out and divided by the single-nucleon response \mathcal{R}_L , one obtains

$$F(q, \omega) = \frac{2\pi m_N^2}{q} \iint_{\Sigma_{RFG}} dp d\mathcal{E} \frac{p}{E_p} S^{RFG}(p, \mathcal{E}), \quad (14)$$

where the domain Σ_{RFG} is defined by the curves in Eq. (5) using Eq. (10). Performing the integration, which can be done analytically, and dividing by the factor

$$\Lambda = \frac{1}{k_F} \left(\frac{m_N}{q} \right) \left(\frac{2m_N T_F}{k_F^2} \right) \quad (15)$$

(note that the quantity $2m_N T_F/k_F^2$ is very close to unity), one ends up with the RFG superscaling function [1]

$$f(\psi) = \frac{3}{4} (1 - \psi^2) \theta(1 - \psi^2) , \quad (16)$$

where

$$\psi = \frac{1}{\sqrt{\xi_F}} \frac{\lambda - \tau}{\sqrt{(1 + \lambda)\tau + \kappa\sqrt{\tau(1 + \tau)}}} \quad (17)$$

with $\xi_F = T_F/m_N$. Here ψ is the scaling variable specifically obtained within the context of the RFG, although it has also been widely employed in more general analyses.

B. The BCS-inspired model

In the spirit of BCS theory we assume that the ground state is now given by

$$|BCS\rangle = \prod_k (u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger) |0\rangle , \quad (18)$$

$|0\rangle$ being the true vacuum, with

$$|u_k|^2 + |v_k|^2 = 1 , \quad (19)$$

to insure the proper normalization of the state, namely

$$\langle BCS | BCS \rangle = \prod_k (|u_k|^2 + |v_k|^2) = 1 . \quad (20)$$

With the assumption in Eq. (18) we have a covariant approximation to the nuclear ground-state wavefunction. We have required that the added pairs always occur with back-to-back momenta (hence the net linear momentum of the system in its rest frame is still zero) and with opposite helicities (hence the net spin of the ground state is zero; we consider only even-even nuclei for simplicity). The creation operators add particles with relativistic on-shell spinors $u(k, \lambda)$.

The state $|BCS\rangle$ does not correspond to a fixed number of particles; in fact it is associated with the spontaneous breaking of the $U(1)$ symmetry in the system. Therefore, it is not an eigenstate of the operator

$$\hat{n}(k) = \sum_s a_{ks}^\dagger a_{ks} , \quad (21)$$

although its expectation value can be computed, yielding

$$n(k) = \langle BCS | \hat{n}(k) | BCS \rangle = |v_k|^2 . \quad (22)$$

We can thus require the particle number A to be conserved on the average, which implies the condition

$$\sum_k |v_k|^2 = A . \quad (23)$$

Concerning the energy, we view our system as being constructed in terms of independent quasi-particles, which indeed is the case for the state in Eq. (18) in leading order, writing accordingly

$$E_{BCS} = \langle BCS | \hat{H} | BCS \rangle = \langle BCS | \sum_{ks} E_k a_{ks}^\dagger a_{ks} | BCS \rangle = \sum_{ks} E_k |v_k|^2 . \quad (24)$$

For sake of simplicity in the following we make the assumption

$$E_k = \sqrt{m_N^2 + k^2} , \quad (25)$$

which reflects the fact that our present goal is to obtain a reasonable variational wave function (the BCS one), not for the purpose of getting the best energy, but rather to provide a way to evaluate the role of correlations on superscaling.

We now face the question of how to compute the wave functions of the daughter nucleus. For this we assume

$$|D(p)\rangle = \frac{1}{\sqrt{\mathcal{N}_p}} a_{p\uparrow} |BCS'\rangle , \quad (26)$$

where \mathcal{N}_p is a normalization factor, and

$$|BCS'\rangle = \prod_k [u'_k(p) + v'_k(p) a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger] |0\rangle , \quad (27)$$

where the coefficients $u'_k(p)$ and $v'_k(p)$ are in general different from those in Eq. (18), but still obey Eq. (19). This point is of crucial relevance for our model, as we shall see below. Note that the daughter nucleus states are labeled by the momentum, \mathbf{p} . For the normalization factor in Eq. (26) we get

$$\mathcal{N}_p = |v'_p(p)|^2 \quad (28)$$

and the expectation value of the operator in Eq. (21) (the system's momentum distribution) turns out to be (see Appendix A)

$$n_{D(p)}(k) = \langle D(p) | \sum_s a_{ks}^\dagger a_{ks} | D(p) \rangle = |v'_k(p)|^2 (1 - \delta_{kp}) . \quad (29)$$

Clearly the condition on the particle number will now read

$$\sum_{k \neq p} |v'_k(p)|^2 = \sum_k |v'_k(p)|^2 - |v'_p(p)|^2 = A - 1 , \quad (30)$$

valid for all of the daughter states, namely for all p . Taking into account Eq. (23), this yields the relation

$$\sum_k |v_k|^2 - \sum_k |v'_k(p)|^2 = 1 - |v'_p(p)|^2 , \quad (31)$$

to be exploited below.

The energy of the daughter nucleus in our model is

$$E_{D(p)} = \langle D(p) | \hat{H} | D(p) \rangle = \sum_{k \neq p} E_k |v'_k(p)|^2 , \quad (32)$$

which, using Eqs. (24) and (31), can be recast as follows:

$$E_{D(p)} = (E_{BCS} - m_N) - T_p |v_p|^2 - \sum_{k \neq p} T_k [|v_k|^2 - |v'_k(p)|^2] . \quad (33)$$

We can then proceed to compute the daughter nucleus spectral function

$$S^{BCS}(p, \mathcal{E}) = |\langle D(p) | a_{p\uparrow} | BCS \rangle|^2 \delta \left[\mathcal{E} - \left(E_{D(p)} - E_0^{(A-1)} \right) \right] \frac{V_A}{A(2\pi)^3} , \quad (34)$$

where $E_0^{(A-1)}$ is the energy $E_{D(p)}$ of the daughter nucleus evaluated at that value of p where it reaches its minimum, to be referred to as k_F in the BCS model. Hence we have

$$\begin{aligned} \mathcal{E}(p) &= E_{D(p)} - E_{D(k_F)} \\ &= T_F |v_{k_F}|^2 - T_p |v_p|^2 + \sum_{k \neq k_F} T_k [|v_k|^2 - |v'_k(k_F)|^2] - \sum_{k \neq p} T_k [|v_k|^2 - |v'_k(p)|^2] . \end{aligned} \quad (35)$$

Later we shall show that the last two terms on the right-hand side of the above provide only small corrections and so can be dropped. We thus see that our model provides a nice variation of the RFG expression in Eq. (12); see later where results are displayed. The matrix element in Eq. (34) can be straightforwardly computed (see Appendix B) and one obtains

$$\begin{aligned}\mathcal{M}_p(p) &= |\langle D(p) | a_{p\uparrow} | BCS \rangle|^2 \\ &= v_p \sqrt{\frac{v'_p(p)}{v'_p(p)^*}} \prod_{k \neq p} [u'_k(p)^* u_k + v'_k(p)^* v_k] .\end{aligned}\quad (36)$$

Thus we end up with the expression

$$\begin{aligned}S^{BCS}(p, \mathcal{E}) &= |\mathcal{M}_p(p)|^2 \delta[\mathcal{E} - \mathcal{E}(p)] \frac{V_A}{A(2\pi)^3} \\ &= |v_p|^2 \delta[\mathcal{E} - \mathcal{E}(p)] \left| \prod_{k \neq p} [u'_k(p)^* u_k + v'_k(p)^* v_k] \right|^2 \frac{V_A}{A(2\pi)^3}\end{aligned}\quad (37)$$

for the daughter nucleus spectral function.

Finally, in order to calculate the longitudinal response function in Eq. (1) what remains to be specified is the integration region Σ , which in turn requires knowledge of the separation energy in Eq. (7). In the present model the latter turns out to be

$$(E_s)_{BCS} = -T_F |v_{k_F}|^2 - \sum_{k \neq k_F} T_k [|v_k|^2 - |v'_k(k_F)|^2] . \quad (38)$$

The question we are now faced with is how to fix the coefficients u_k , v_k , $u'_k(p)$ and $v'_k(p)$ entering in the wave functions of the initial ground state and final daughter nucleus state. As a first step in this direction let us recall the normalization constraints that these coefficients have to fulfill and deduce expressions for them in the thermodynamic limit $A \rightarrow \infty$, $V_A \rightarrow \infty$, $A/V_A = \rho_A$. For the initial nucleus one has

$$\lim_{V_A} \frac{1}{V_A} \sum_k |v_k|^2 = \int \frac{d^3k}{(2\pi)^3} |v(k)|^2 = \lim_{V_A} \frac{A}{V_A} = \rho_A , \quad (39)$$

while for the daughter nucleus one obtains

$$\begin{aligned}\lim_{V_{A-1}} \frac{1}{V_{A-1}} \left[\sum_k |v'_k(p)|^2 - |v'_p(p)|^2 \right] &= \int \frac{d^3k}{(2\pi)^3} |v'(k; p)|^2 - \left(\rho_A \frac{V_A}{V_{A-1}} - \rho_{A-1} \right) |v'(p; p)|^2 \\ &= \lim_{V_{A-1}} \frac{A-1}{V_{A-1}} = \rho_{A-1} .\end{aligned}\quad (40)$$

Note that if Eq. (39) is valid then the spectral function in Eq. (37) is correctly normalized, namely $\int d^3p d\mathcal{E} S^{BCS}(p, \mathcal{E}) = 1$, provided that $\prod_{k \neq p} [u'_k(p)^* u_k + v'_k(p)^* v_k] = 1$.

The next constraint for the daughter nucleus wave function's coefficients stems from the stability condition, which requires that the energy of the latter fulfill the relation

$$\left. \frac{dE_{D(p)}}{dp} \right|_{p=k_F} = 0 , \quad (41)$$

which, as already mentioned, defines the Fermi momentum k_F for the $(A-1)$ system. Neglecting the last term on the right-hand side of Eq. (33) this becomes

$$\left. \frac{d}{dp} [T(p) |v(p)|^2] \right|_{p=k_F} = 0 , \quad (42)$$

which can be recast as follows

$$|v(k_F)|^2 \left. \frac{dT(p)}{dp} \right|_{p=k_F} + T(k_F) v(k_F)^* \left. \frac{dv(p)}{dp} \right|_{p=k_F} + T(k_F) v(k_F) \left. \frac{dv(p)^*}{dp} \right|_{p=k_F} = 0 . \quad (43)$$

Assuming that v is real, this becomes

$$v(k_F) \frac{k_F}{\sqrt{k_F^2 + m_N^2}} + 2 \left(\sqrt{k_F^2 + m_N^2} - m_N \right) \frac{dv(p)}{dp} \Big|_{p=k_F} = 0 . \quad (44)$$

In accord with the BCS model and with the physics we intend to explore in the next section, we now choose the following three-parameter expression

$$v^2(k) = \frac{c}{e^{\beta(k-\tilde{k})} + 1} \quad (45)$$

for the v coefficients. Note that when $\beta \rightarrow \infty$ one must recover the RFG. Substituting Eq. (45) into Eq. (44) we obtain

$$k_F \left[e^{\beta(k_F - \tilde{k})} + 1 \right] = \beta \sqrt{k_F^2 + m_N^2} \left(\sqrt{k_F^2 + m_N^2} - m_N \right) e^{\beta(k_F - \tilde{k})} , \quad (46)$$

which can be solved for \tilde{k} in terms of k_F and β , yielding

$$\tilde{k} = k_F + \frac{1}{\beta} \log \left[\frac{\beta}{k_F} \sqrt{k_F^2 + m_N^2} \left(\sqrt{k_F^2 + m_N^2} - m_N \right) - 1 \right] . \quad (47)$$

The inverse of this equation then gives k_F in terms of \tilde{k} and β .

We use next the normalization condition in Eq. (39) to fix the parameter c in Eq. (45). We obtain

$$c(\beta, \tilde{k}) = - \frac{\pi^2 \beta^3 \rho_A}{Li_3(-e^{\beta \tilde{k}})} , \quad (48)$$

where

$$Li_3(z) = \frac{1}{2} \int_0^\infty \frac{k^2}{e^k/z - 1} dk \quad (49)$$

is the trilogarithmic function.

For the daughter nucleus we employ the same form of parametrization as in Eq. (45), namely

$$v'^2(k; p) = \frac{c_D}{e^{\beta_D[k - \tilde{k}_D(p)]} + 1} \quad (50)$$

and require the parameters to fulfill the normalization condition for the $A - 1$ system, namely, Eq. (40). Assuming $\rho_{A-1} = \rho_A \equiv \rho$ and $V_A/V_{A-1} = A/(A - 1)$ this leads to the equation

$$\frac{c}{\pi^2 \beta^3} Li_3(-e^{\beta \tilde{k}}) + \frac{c_D}{\pi^2 \beta_D^3} Li_3(-e^{\beta_D \tilde{k}_D(p)}) = \frac{\rho}{A - 1} \times \frac{c_D}{e^{\beta_D[p - \tilde{k}_D(p)]} + 1} . \quad (51)$$

In the thermodynamic limit the right-hand side vanishes and Eq. (51) is simply solved by $c_D = c$, $\beta_D = \beta$, $\tilde{k}_D(p) = \tilde{k}$; hence $v'^2(k; p) = v^2(k)$. As anticipated, this implies that the last two terms in Eq. (35) vanish and that the product in Eq. (37) is unity, so that the spectral function of the daughter nucleus is correctly normalized to unity. It must be emphasized that the coefficients v and v' become identical in the thermodynamic limit, but are different for finite A . Hence it is crucial to compute the dispersion relation $\mathcal{E}(p)$ when A is finite and *then* take the thermodynamic limit. This is reminiscent of Koopmans' theorem in Hartree-Fock theory [24].

As far as the parameter β is concerned, it clearly controls both the modifications of the momentum distribution near the Fermi surface (promotion of pairs due to residual NN interactions, both long- and short-range) and also the tail of the momentum distribution due to short-range NN correlations. Indeed, for β very large one recovers the familiar θ -distribution, while for smaller and smaller β more and more particles are pulled out of the Fermi sea and produce a significant tail for the momentum distribution at large momenta. The impact of the physics expressed by the parameter β on the superscaling function is explored in the next section.

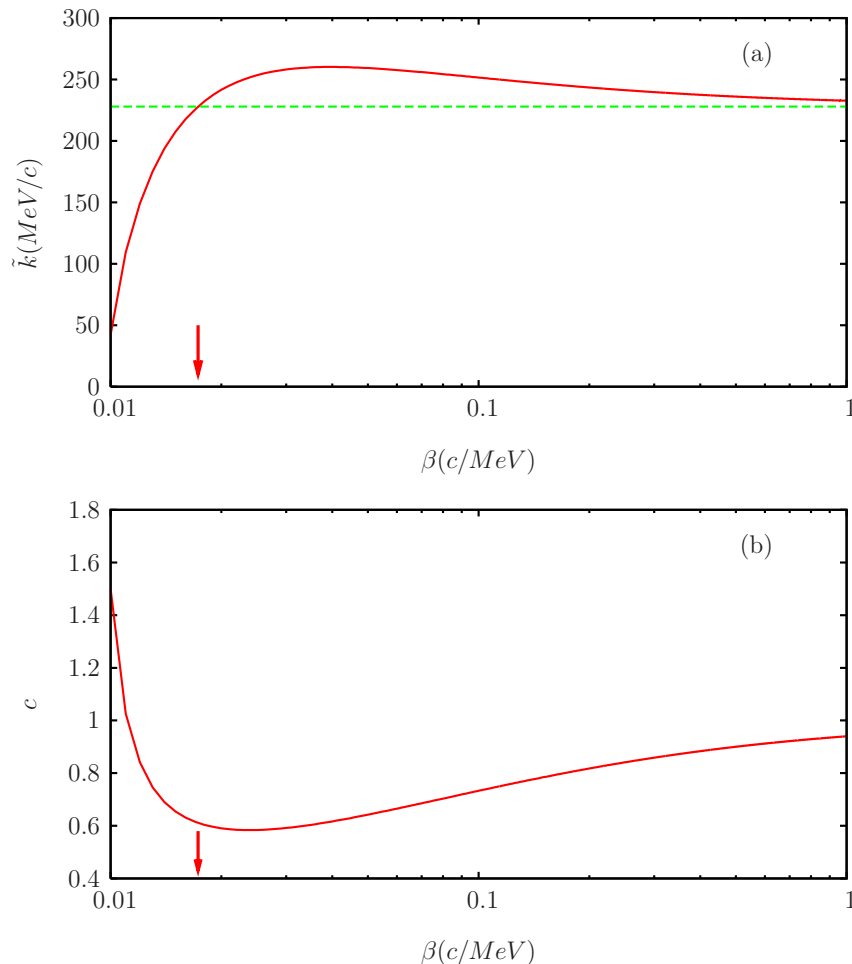


FIG. 1: (Color online) Panel (a): The parameter \tilde{k} , given in Eq. (47), shown as a function of β . The straight line indicates the Fermi momentum $k_F = 228$ MeV/c. Panel (b): The parameter c , given in Eq. (48), plotted versus β for $\rho_A = k_F^3/(6\pi^2)$. In both panels the arrow indicates the critical value $\beta_{\text{crit}} = 0.017$ c/MeV.

IV. NUMERICAL RESULTS

The model is specified by three parameters, \tilde{k} , β and c , the last being fixed by the normalization condition in Eq. (48). The Fermi momentum in this model is determined by the daughter nucleus stability condition in Eq. (42). Alternatively, as it is somewhat easier to achieve, one can fix the Fermi momentum k_F and use Eq. (47) to determine \tilde{k} . In presenting the results obtained using our model it is then convenient to start by displaying the behaviour of the parameters \tilde{k} and c , which are fixed by the physical conditions of normalization and stability, versus β for given k_F . When \tilde{k} , c and β are known so are the wave functions of the initial and final nuclei.

In Fig. 1 the parameters \tilde{k} and c are plotted versus β . They stay constant (in fact the almost constant value of \tilde{k} is quite close to the value $k_F = 228$ MeV/c we use as an input) until a critical value $\beta_{\text{crit}} = 0.017$ c/MeV is reached where \tilde{k} (c) displays a dramatic decrease (increase). This value corresponds to the change of sign of the logarithmic term in Eq. (47), namely

$$\beta_{\text{crit}} = \frac{2k_F}{T_F(T_F + m_N)} . \quad (52)$$

For values of β lower than β_{crit} the nuclear momentum distribution becomes very much extended beyond the Fermi sphere associated with the input value of k_F ; we shall return below to discuss this delicate issue. Thus our results appear to point to the existence of a narrow domain of β around β_{crit} , below which the system becomes strongly disrupted by correlations. This has a strong impact on the structure of the superscaling function, as we shall see later. As stated above, for the results shown in the figures we have taken $\rho = k_F^3/(3\pi^2)$ and $k_F = 228$ MeV/c (a

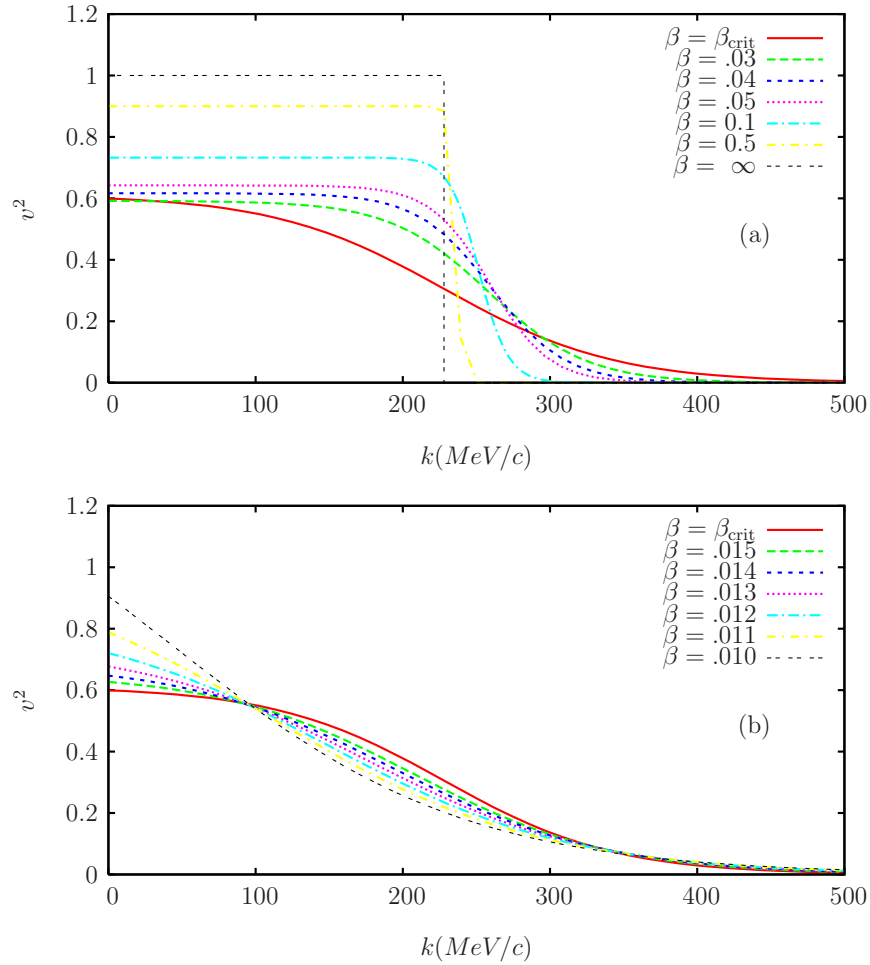


FIG. 2: (Color online) Momentum distribution of the initial state, Eq. (45), evaluated for $k_F = 228$ MeV/c, $\rho_A = k_F^3/(6\pi^2)$ and different values of β (in c/MeV) above (a) and below (b) the critical value β_{crit} .

value appropriate for ^{12}C), although our results appear to be quite insensitive to k_F as long as it is kept within the range appropriate for the physics of atomic nuclei. We display the momentum distribution of the initial nucleus in Fig. 2 for a few values of β larger (panel a) or smaller (panel b) than β_{crit} . The progressive development of a tail in the momentum distribution is clearly seen in the figure.

The next important issue to be addressed is to determine where the spectral function is nonzero in the (\mathcal{E}, p) plane. The answer is found in Fig. 3 where the support of the spectral functions of the RFG and of our BCS-inspired model are displayed and compared for $\beta=0.1$ c/MeV and the same k_F used in Fig. 1. Both spectral functions of course are just δ -functions, but concerning their support two major differences distinguish the two:

1. In the range of momenta where both exist the excitation spectrum of the daughter system is substantially softer than the RFG one;
2. For missing momenta larger than k_F the BCS case, unlike the RFG, continues to display a spectrum (its spectral function continues to have a support) which in the thermodynamic limit rises quite suddenly with p until it reaches the value \mathcal{E} assumes for vanishing missing momentum, namely

$$\mathcal{E}_{\text{max}} = T_F |v(k_F)|^2. \quad (53)$$

This energy, which can be shown to be the upper limit of the daughter spectrum and which cannot be exceeded because of the stability condition in Eq. (41), is reached only at $p = \infty$, but over a large span of momenta \mathcal{E} remains almost constant, thus corresponding to the situation of an eigenvalue with infinite degeneracy stemming from the symmetry $U(1)$ associated with the particle number conservation. As p is lowered, approaching the Fermi surface, the degeneracy is lifted and we face a situation of a spontaneously broken symmetry, reflected in

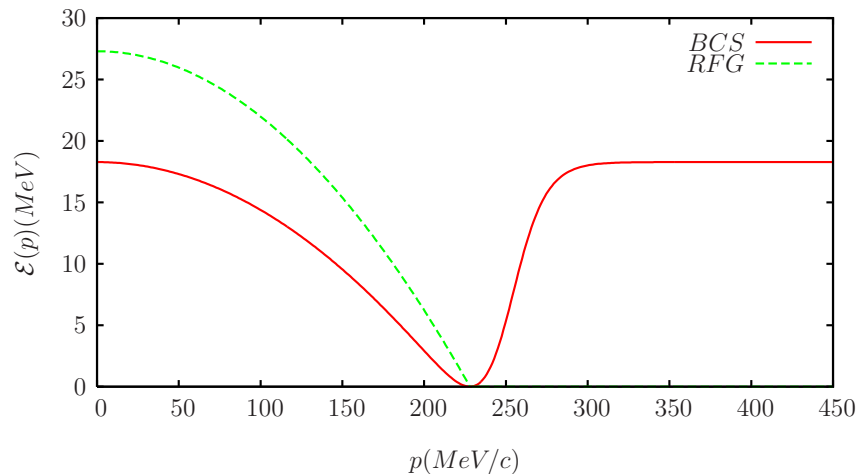


FIG. 3: (Color online) The excitation energy \mathcal{E} computed according to Eq. (35) neglecting the last two terms for $k_F = 228$ MeV/c, $\rho_A = k_F^3/(6\pi^2)$, $\beta = 0.1$ c/MeV. The RFG results are also shown for comparison.

the structure of our state which contains components of all possible particle number. This situation is strongly reminiscent of superconductivity, where the spontaneous symmetry breaking also occurs in the proximity of the Fermi surface (in this case, however, the control parameter is the temperature; in our case it is the momentum itself).

This set of degenerate states has a dramatic impact on the superscaling function of our model, which we obtain using the procedure already outlined for the RFG in Section III employing the same dividing factor Λ given in Eq. (15). The function f is displayed versus the scaling variable ψ in Eq. (17) for a few values of β and q in Fig. 4. For comparison the RFG result in Eq. (16) and the averaged experimental data [7, 25] are also shown. One sees that to get f for large positive ψ we have to integrate in the (\mathcal{E}, p) plane in domains (see CDM) encompassing large fractions of those degenerate states discussed above. These are thus the cause of the asymmetry of the scaling function with respect to $\psi = 0$ appearing in Fig. 4. For ψ large and negative these states are to a large extent excluded from entering into the building up of f . The fact that this effect is more and more pronounced as β becomes smaller reflects the impact of the tail of the momentum distribution which indeed grows when β decreases and, as a consequence, more degenerate states participate to build up f . Note that values of β around the critical value ($\beta \simeq 0.01 - 0.02$ c/MeV) yield a tail which is in qualitative agreement with the experimental data.

As far as scaling of the first kind is concerned, Fig. 4 shows that this is quickly reached in the vicinity of the QEP, although not so to the right and to the left of it. A closer examination of the results (see Figs. 5 and 6, where f is plotted on a logarithmic scale for a wider q -range at $\beta=0.01$ c/MeV) shows, however, that also here the BCS model does scale, however with an onset reached only for $q \simeq 1.5$ GeV/c, namely for larger momenta than when at the QEP where the onset already occurs at about 500 MeV/c. Also from Figs. 5 and 6 it appears that the scaling regime is reached faster to the right than to the left of the QEP. Moreover, the asymptotic value for $\psi < 0$ is approached from below, namely the superscaling function grows with q until it reaches its asymptotic value, in contrast with the experimental findings. This reflects the fact that our model, although appealingly simple, is not able to account for features of this kind. We conjecture that a non-uniform strength function should be associated with the continuum spectrum of the daughter nucleus, which in general implies the development of a more elaborate version of our model. For instance, a further extension of our model could allow the coefficients v to become complex to account for the finite lifetime of the excited states. Note that the same trend of approaching first-kind scaling from below is also found in [26, 27, 28] within the framework of the Coherent Density Fluctuation Model where realistic nucleon momentum and density distributions are used [26, 27, 28]. On the other hand, in relativistic mean-field theory [10] the approach is from above, and thus in better accord with the experimental data.

Finally, using the present BCS model, we investigate the second-kind scaling behaviour by repeating our calculations for different values of the external parameter k_F , each one taken to represent a specific nucleus. The results are shown in Fig. 7, where it appears that the BCS model breaks scaling of the second kind both at the QEP and for negative ψ , whereas it fulfills it at high positive values of the scaling variable. However, this is not how the original idea of scaling of the second kind was introduced [5, 6]. In that study of experimental data a momentum k_A (actually called k_F in that work, although it was in fact a phenomenological parameter and not necessarily the “true” Fermi momentum as it must reflect both initial- and final-state interaction effects) was chosen for each nuclear species and used in the

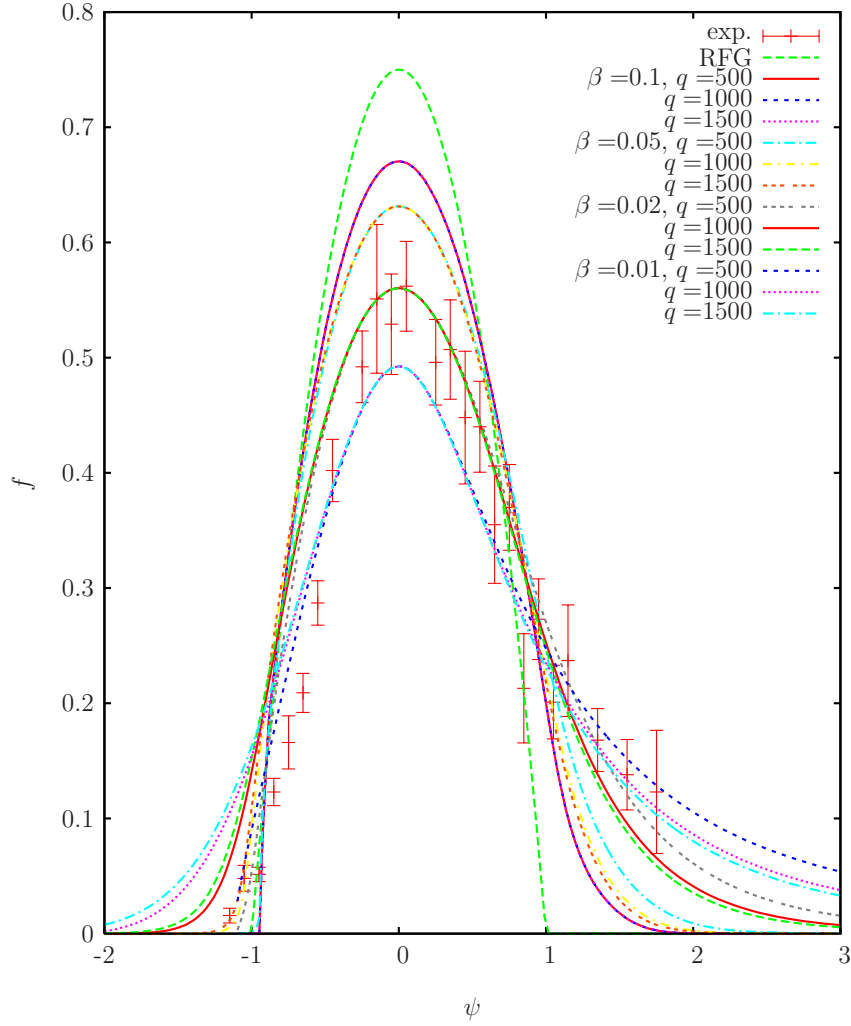


FIG. 4: (Color online) The superscaling function f defined in Eq.(14) plotted versus the scaling variable (17) in the RFG model and in the present BCS model for three values of q (in MeV/c) and different values of β (in c/MeV). As usual, $k_F = 228$ MeV/c. Data are taken from [7, 25].

definition in Eq. (17) of the scaling variable ψ and of the dividing factor in Eq. (15), which will now be written as

$$\Lambda_A = \frac{1}{k_A} \left(\frac{m_N}{q} \right) \left(\frac{2m_N T_F}{k_F^2} \right). \quad (54)$$

For simplicity, in the present approach the value of k_A is chosen in order to have all the corresponding superscaling functions coincide at the QEP, thus realizing superscaling at least where the nuclear response is the largest. We display the results in Fig. 8, where each curve corresponds to given k_F and k_A , for $q = 1000$ MeV/c and $\beta = 0.01$ c/MeV. In this case it turns out that scaling of second kind at the QEP can indeed be achieved using the following empirical expression

$$\frac{k_A}{k_F} = 1 + \alpha(k_0 - k_F), \quad (55)$$

where k_0 is just a parameter (here $k_0 \equiv k_F = 228$ MeV/c) chosen to fix the height of the peak and where $\alpha = 6 \times 10^{-4}$ c/MeV. That is, k_A is found to fall very slightly with increasing k_F for realistic values of the Fermi momentum. Over much of the range of ψ shown in the figure one sees relatively good second-kind scaling, although the results still point to a sizable violation of the second kind scaling in the scaling domain (large negative scaling variable). Finally, we note that for higher values of β the parameter α is found to decrease, as it should, since the RFG result $\alpha = 0$ must be recovered when $\beta \rightarrow \infty$. The behaviour of α versus the parameter β is shown in Fig. 9.

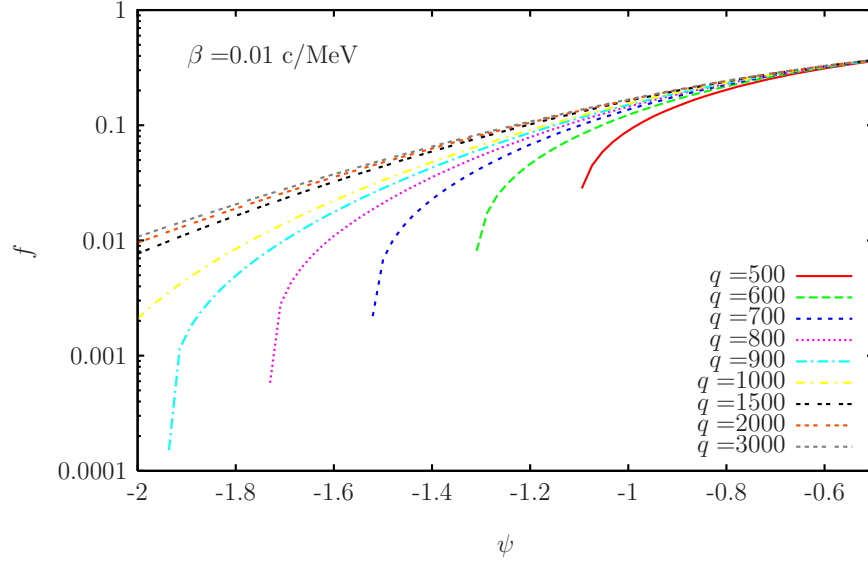


FIG. 5: (Color online) The superscaling function f in the negative ψ region plotted for several values of q (in MeV/c) and $\beta = 0.01$ c/MeV. As usual, $k_F = 228$ MeV/c.

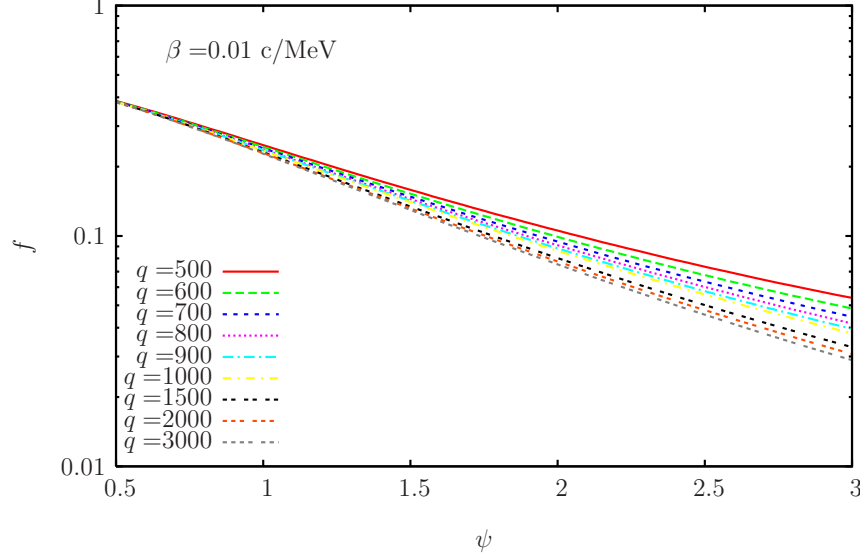


FIG. 6: (Color online) The superscaling function f in the positive ψ region plotted for several values of q (in MeV/c) and $\beta = 0.01$ c/MeV. As usual, $k_F = 228$ MeV/c.

V. CONCLUSIONS

In the present study a simple extension of the relativistic Fermi gas model for studies of relatively high-energy inclusive electroweak cross sections has been developed. Starting from the RFG in which a degenerate gas of nucleons is assumed for the nuclear ground state, in this extension pairs of particles are promoted from below the Fermi surface to above, yielding a spectral function and the resulting momentum distribution with Fourier components for all values of momentum. In the spirit of the RFG this new model has been constructed in a way that maintains covariance — the pairs of particles involved all have spinor products with opposing momenta and helicities, thereby producing a ground state with net momentum zero and net spin zero.

Specifically, the model employed is inspired by the BCS approach used in treating superconductivity, both in condensed matter physics and in the nuclear physics problem of superconducting nuclei, in which an infinite product of operators of the form $u_p + v_p a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger$ with quasiparticle weighting factors u_p and v_p is assumed. The specific choice of these weighting factors characterizes the particular model assumed in the present study, namely, such that the

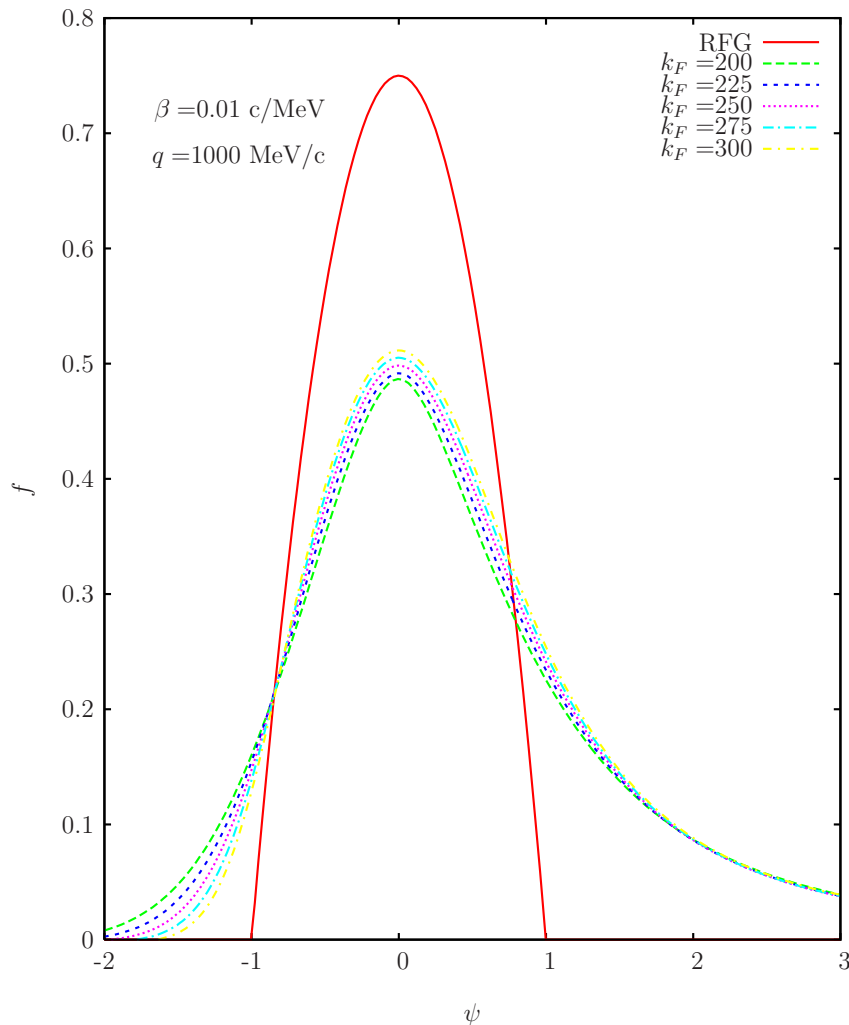


FIG. 7: (Color online) The superscaling function f plotted versus ψ for several values of the Fermi momentum k_F (in MeV/c) at $\beta = 0.01$ c/MeV and $q = 1000$ MeV/c.

ground-state momentum distribution has a rounded Fermi surface and a long tail to represent effects from both long- and short-range correlations. It is, however, worth recalling that our state only accounts for correlations between pairs of fermions in time-reversed states, namely with opposite momenta, as occur when employing the pairing Hamiltonian. Clearly the parametrization used in the present study is only one choice for the weighting factors and it is straightforward to employ other more realistic forms when warranted (see for instance [29, 30, 31]). The nuclear particle number and total energy of the ground state are obtained as expectation values using this form and constrained to chosen values A and M_A . The final state (the one resulting after the electroweak interaction with the ground state has taken place) is chosen to be a plane-wave outgoing nucleon plus a daughter nucleus recoiling in a way that conserves momentum. In the rest frame of a general daughter state the particle number and total energy may be found and accordingly the daughter ground state may be identified. With these ingredients in hand it is then straightforward to obtain the nuclear spectral function and from it to compute the rest of the desired functions, namely, the nuclear responses (in the present work only the EM longitudinal response is discussed, although it is clearly straightforward to obtain any electroweak responses using the same ideas), and the superscaling function.

With respect to the last, the results presented in Section IV show that the model is capable of accounting for several, but not all, of the features seen in other studies of superscaling, both phenomenologically and within the context of alternative modeling. Specifically, scaling of the first kind (independence of q) is observed for momentum transfers that are high enough. Near the quasielastic peak the onset of first-kind scaling is quite rapid, whereas away from the peak it is somewhat slower, that is, requires a high value of q to occur. As with most models (the exception being relativistic mean-field theory and its derivatives) the approach to first-kind scaling is from below, whereas the data in the scaling region approach from above. Finally, second-kind scaling (independence of nuclear species) is seen

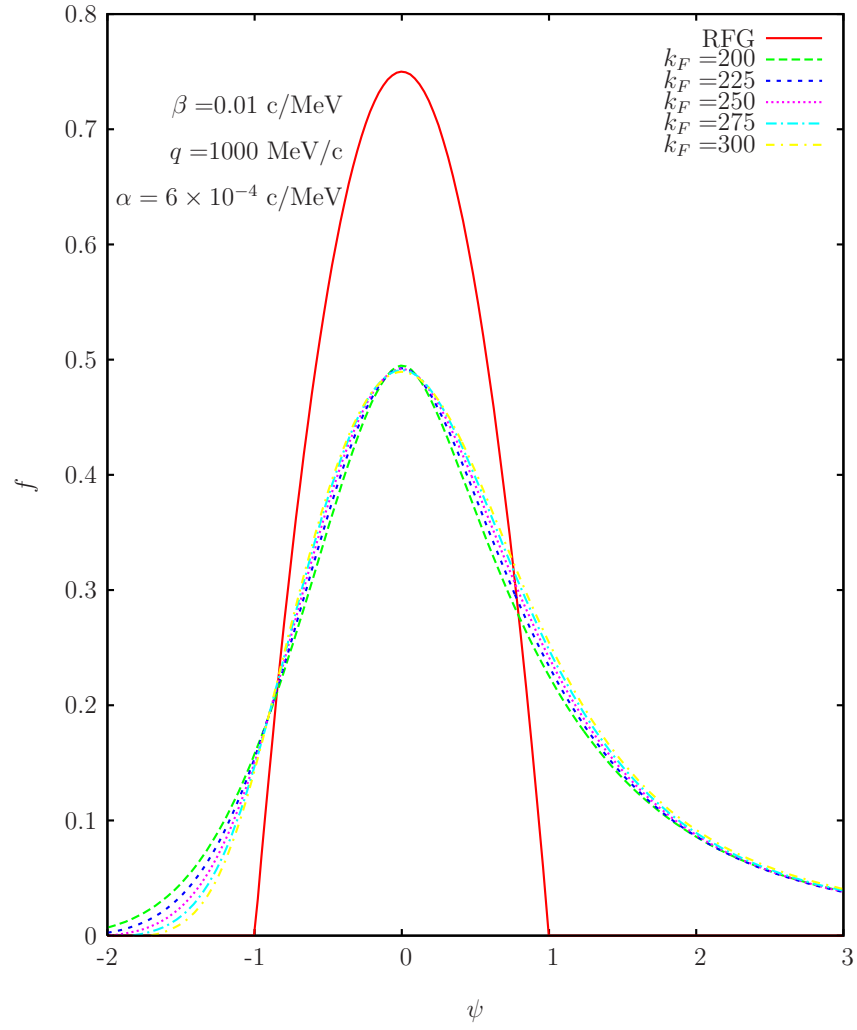


FIG. 8: (Color online) The superscaling function f plotted versus ψ for several values of the Fermi momentum k_F (in MeV/c) and with k_A devised in such a way that the peaks coincide (see text).

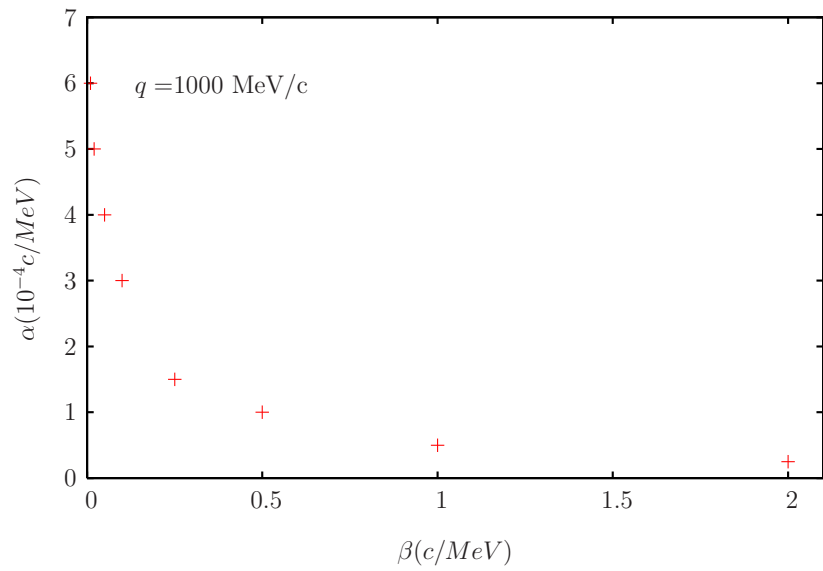


FIG. 9: (Color online) The parameter α of Eq. (55) shown as a function of β for $q = 1000$ MeV/c.

to be good although some scaling violations are observed.

APPENDIX A: MOMENTUM DISTRIBUTION OF THE DAUGHTER NUCLEUS

The momentum distribution associated with the daughter nucleus reads

$$\begin{aligned}
 n_{D(q)}(k) &= \langle D(q) | \sum_s a_{ks}^\dagger a_{ks} | D(q) \rangle = \\
 &= \frac{1}{|v'_q(q)|^2} \sum_s \left\{ \langle 0 | [u'_q(q)^* + v'_q(q)^* a_{-q\downarrow} a_{q\uparrow}] [u'_k(q)^* + v'_k(q)^* a_{-k\downarrow} a_{k\uparrow}] a_{q\uparrow}^\dagger a_{ks}^\dagger \times \right. \\
 &\quad \prod_{p \neq q \neq k} [u'_p(q)^* + v'_p(q)^* a_{-p\downarrow} a_{p\uparrow}] [u'_p(q) + v'_p(q) a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger] a_{ks} a_{q\downarrow} \\
 &\quad \left. [u'_k(q) + v'_k(q) a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger] [u'_q(q) + v'_q(q) a_{q\uparrow}^\dagger a_{-q\downarrow}^\dagger] | 0 \rangle \right\} = \\
 &= \frac{1}{|v'_q(q)|^2} \left\{ |v'_q(q)|^2 |v'_k(q)|^2 \langle 0 | \prod_{p \neq q \neq k} [u'_p(q)^* + v'_p(q)^* a_{-p\downarrow} a_{p\uparrow}] [u'_p(q) + v'_p(q) a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger] | 0 \rangle \right\} \\
 &= |v'_k(q)|^2 (1 - \delta_{kq}). \quad (\text{A1})
 \end{aligned}$$

APPENDIX B: EVALUATION OF THE MATRIX ELEMENT

Considering the matrix element

$$\mathcal{M}_p(q) = \langle D(q) | a_{p\uparrow} | BCS \rangle \quad (\text{B1})$$

we observe that

$$\begin{aligned}
 a_{p\uparrow} | BCS \rangle &= a_{p\uparrow} \prod_k (u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger) | 0 \rangle \\
 &= \left[\prod_{k \neq p} (u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger) \right] a_{p\uparrow} (u_p + v_p a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger) | 0 \rangle \\
 &= \left[\prod_{k \neq p} (u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger) \right] v_p a_{-p\downarrow}^\dagger | 0 \rangle \\
 &= v_p a_{-p\downarrow}^\dagger \prod_{k \neq p} (u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger) | 0 \rangle \\
 &= v_p a_{-p\downarrow}^\dagger | BCS(\neq p) \rangle, \quad (\text{B2})
 \end{aligned}$$

having defined

$$| BCS(\neq p) \rangle = \prod_{k \neq p} (u_k + v_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger) | 0 \rangle, \quad (\text{B3})$$

the BCS wavefunction without the term p . Similarly

$$| D(q) \rangle = \frac{1}{\sqrt{|v'_q(q)|^2}} a_{q\uparrow} | BCS' \rangle = \sqrt{\frac{v'_q(q)}{v'_q(q)^*}} a_{-q\downarrow}^\dagger | BCS'(\neq q) \rangle \quad (\text{B4})$$

and therefore

$$\mathcal{M}_p(q) = v_p \sqrt{\frac{v'_q(q)}{v'_q(q)^*}} \langle BCS'(\neq q) | a_{-q\downarrow} a_{-p\downarrow}^\dagger | BCS(\neq p) \rangle$$

$$= v_p \sqrt{\frac{v'_q(q)}{v'_q(q)^*}} \left(\delta_{pq} < BCS'(\neq q) | BCS(\neq p) > - < BCS'(\neq q) | a_{-p\downarrow}^\dagger a_{-q\downarrow} | BCS(\neq p) > \right) . \quad (B5)$$

Now

$$\delta_{pq} < BCS'(\neq q) | BCS(\neq p) > = \delta_{pq} \prod_{k \neq q} [u'_k(q)^* u_k + v'_k(q)^* v_k] , \quad (B6)$$

whereas the second term in Eq. (B5) vanishes, since

$$a_{-q\downarrow} | BCS(\neq p) > = 0 \quad \text{if } p = q \quad (B7)$$

and

$$a_{-q\downarrow} | BCS(\neq p) > = -v_q a_{q\uparrow}^\dagger | BCS(\neq p, \neq q) > \quad \text{if } p \neq q, \quad (B8)$$

namely

$$a_{-q\downarrow} | BCS(\neq p) > = (\delta_{pq} - 1) v_q a_{q\uparrow}^\dagger | BCS(\neq p, \neq q) > \quad (B9)$$

and, similarly

$$a_{-p\downarrow} | BCS'(\neq q) > = (\delta_{pq} - 1) v'_p(q) a_{p\uparrow}^\dagger | BCS'(\neq p, \neq q) > . \quad (B10)$$

Then

$$\begin{aligned} < BCS'(\neq q) | a_{-q\downarrow} a_{-p\downarrow}^\dagger | BCS(\neq p) > \\ &= (\delta_{pq} - 1) v'_p(q)^* v_q < BCS'(\neq p, \neq q) | a_{p\uparrow} a_{q\uparrow}^\dagger | BCS(\neq p, \neq q) > = 0 . \end{aligned} \quad (B11)$$

Eqs.(B6) and (B11), when inserted into Eq. (B5), finally yield

$$\mathcal{M}_p(q) = \delta_{pq} v_q \sqrt{\frac{v'_q(q)}{v'_q(q)^*}} \prod_{k \neq q} [u'_k(q)^* u_k + v'_k(q)^* v_k] . \quad (B12)$$

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